

On Krammer's Representation of the Braid Group

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Abstract

A connection is made between the Krammer representation and the Birman-Murakami-Wenzl algebra. Inspired by a dimension argument, a basis is found for a certain irrep of the algebra, and relations which generate the matrices are found. Following a rescaling and change of parameters, the matrices are found to be identical to those of the Krammer representation. The two representations are thus the same, proving the irreducibility of one and the faithfulness of the other.

1 Introduction

In [BW], Joan Birman and Hans Wenzl constructed a two-parameter family of algebras related to braid groups and the Kauffman knot polynomial, and analyzed its semisimple structure. The same algebras were discovered simultaneously and independently by Jun Murakami in [M]. Since the braid group maps homomorphically into this algebra, the representations on the simple summands of the algebra give irreducible representations of the braid group. We will focus on one particular irreducible representation (for each n) out of this collection, and show that it is identical to an independently discovered representation. This is parallel to a similar identification which has long been known between the Burau representation and a certain summand of the Hecke algebra.

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Daan Krammer, in [K], defined a representation of the braid group using its interpretation as the automorphism group of the punctured disk. Using geometric arguments, he constructed a skein relation between items called forks. From the skein relation, he was able to list a set of algebraic equations describing the action of braid group generators on these forks, some of which form a basis for an invariant module. These equations thus describe matrix entries for the representation.

Using techniques related to the solution of the word problem in the braid groups in [BKL], Krammer was then able to prove that this representation was faithful for $n = 4$, thus showing

Theorem. (Krammer) *B_4 is linear.*

This discovery revolutionized the study of braid group linearity, since for a long time it seemed that the Burau representation was the best candidate to give a faithful representation of the braid group.

Nevertheless, Krammer's construction was not widely publicized until Stephen Bigelow expanded Krammer's result using topological methods:

Theorem. (Bigelow) *Krammer's representation is faithful for all n , thus the braid groups are linear.*

This finally answered an important question in braid theory which had stood since the introduction of the Burau representation in 1935.

The main result of this paper gives a connection between Krammer's representation and the Birman-Murakami-Wenzl (BMW) algebra:

Main Theorem. *Krammer's representation of the braid group B_n is identical to the $(n - 2) \times 1$ irreducible representation of the BMW algebra.*

My proof uses purely algebraic methods, and can be deduced solely from the equations given in [K] for the action of the braid group, and the equations given in [BW] for relations within the algebra.

One immediate consequence of this theorem is

Corollary 1. *Krammer's representation is irreducible.*

And another follows quickly from Bigelow's result:

Corollary 2. *The regular representation of the BMW algebra is faithful.*

Vaughan Jones also discovered this main result, simultaneously and independently. However, his methods are different from mine, and involve a somewhat deeper understanding of the algebraic structure.

The representation which Krammer spelled out explicitly was originally discovered by Ruth Lawrence in [L] as a representation of the Hecke algebra, using homology of configuration spaces similar to those used by Bigelow. This connection relates to a topological interpretation of the Jones polynomial which is still in the process of being explored. With the addition of Krammer's presentation of the representation, we can also make connections to other algebras and knot polynomials.

1.1 Acknowledgements

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2 Background and Definitions

The braid group B_n has many definitions and interpretations. Perhaps the easiest to see is a pictorial one. A *braid* is a diagram consisting of two horizontal bars, one at the top and one at the bottom of the figure, with n nodes on each bar (usually drawn equally spaced), and n *strands*, always running strictly downward, connecting the upper and lower nodes. This figure represents an isotopy class of embeddings of the strands in 3-space, so the strands are allowed (in fact, required) to cross over and under each other rather than intersect, and the directions of these crossings are marked in a conventional way on the diagram. Among all braids, those which are isotopic in $\mathbb{R} \times I$ are identified.

Braids form a group. The identity element is the braid with no crossings, multiplication is concatenation (draw one diagram above the other, and erase the center bar), and the standard set of generators (known as *Artin generators*) consists of braids σ_i ($1 \leq i < n$), where the only crossing is that of the i th strand under the next strand. See Figure 1(a). The relations in the braid group are easily described:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| \geq 2 \quad (1)$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}. \quad (2)$$

One of the long-standing questions about the braid group has been whether it is linear, that is, whether there exists any faithful representation into a matrix group. A common method of constructing representations of the braid group is to map the braid group homomorphically into a finite-dimensional algebra, and use the algebra's regular representation. One such algebra that was used is the Hecke algebra, a deformation of the complex symmetric algebra $\mathbb{C}S_n$, and another is the Birman-Murakami-Wenzl algebra, similarly a deformation of the Brauer algebra.

The BMW algebra $C_n(l, m)$ can be defined on invertible generators G_i ($1 \leq i < n$), which satisfy the braid relations described above, and non-invertible elements E_i defined via the formula

$$G_i + G_i^{-1} = m(1 + E_i). \quad (3)$$

The additional relations are:

$$E_i E_{i\pm 1} E_i = E_i \quad (4)$$

$$G_{i\pm 1} G_i E_{i\pm 1} = E_i G_{i\pm 1} G_i = E_i E_{i\pm 1} \quad (5)$$

$$G_{i\pm 1} E_i G_{i\pm 1} = G_i^{-1} E_{i\pm 1} G_i^{-1} \quad (6)$$

$$G_{i\pm 1} E_i E_{i\pm 1} = G_i^{-1} E_{i\pm 1} \quad (7)$$

$$E_{i\pm 1} E_i G_{i\pm 1} = E_{i\pm 1} G_i^{-1} \quad (8)$$

$$G_i E_i = E_i G_i = l^{-1} E_i \quad (9)$$

$$E_i G_{i\pm 1} E_i = l E_i \quad (10)$$

$$E_i^2 = (m^{-1}(l + l^{-1}) - 1) E_i \quad (11)$$

$$G_i^2 = m(G_i + l^{-1} E_i) - 1. \quad (12)$$

See [BW]. Some of these relations can be deduced from others; however, I am not concerned here with a minimal presentation. For those readers who are, one can be found in [W1].

In addition, it can be deduced that E_i commutes with both E_j and G_j , if $|i - j| \geq 2$.

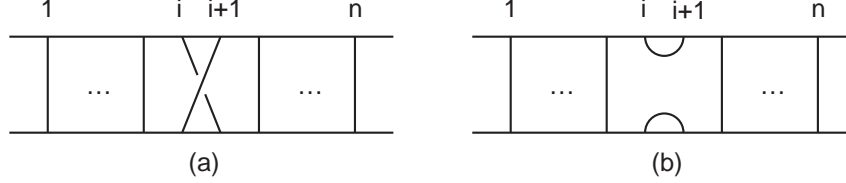


Figure 1: (a) The braid generator σ_i . (b) The braid-like algebra element E_i .

As in the case of the braid group, we can associate several elements of the algebra to pictures. The generators G_i can be identified with the same braid diagrams (Figure 1(a)) described above for σ_i . The elements E_i can be identified with similar diagrams where the i th node on the top is joined to the next node on top, and similarly on the bottom (with all other nodes connected vertically as with G_i). See Figure 1(b). Since the “strands” do not run strictly downward, this is not a braid element. However, we can still “multiply” these diagrams by each other and by braids, to again get isotopy classes of embeddings in 3-space.

The reader may notice that each of the above relations in the algebra which does not involve addition actually represents an isotopy relation of these braid-like elements. Consequently, every monomial in the algebra can be uniquely represented by such a picture, up to isotopy.

The irreducible representations of an algebra such as this one can be identified using its Bratelli diagram, which encodes the decomposition of each semisimple algebra C_n into its simple summands. The regular representation on each summand (which, as a vector space, is an invariant subspace) is then an irreducible representation of the algebra and, by extension, the braid group which maps into it. The connections in the diagram between levels encode the inductions or restrictions of the representations to the next or previous level (by adding or ignoring the final generator).

The Bratelli diagram of the Hecke algebra is the same as the Bratelli diagram of the complex group algebra of the symmetric group. So each simple module, thus each irreducible representation, of H_n is indexed by a Young diagram with n boxes, and connections from a particular module to each succeeding level are to the Young diagrams obtained by adjoining one box to the Young diagram in question. Define notation as follows: let $V_{n,\lambda}$

denote the module or representation in the n th level of the Bratelli diagram, labeled by the Young diagram λ .

The Hecke algebra is a quotient of the BMW algebra (obtained by setting $E_i = 0$). Consequently, it appears as a direct summand of C_n , and its Bratelli diagram is contained in that of the larger algebra. The complete structure of the Bratelli diagram of C_n , as is explained in [W1], is the same as the Brauer algebra D_n :

Theorem. (Wenzl) (a) D_n is semisimple.

(b) The simple components of D_n are labeled by the set of Young diagrams with $n - 2k$ boxes ($k \in \mathbb{Z}^+$).

(c) If $V_{n,\lambda}$ is a simple D_n module it decomposes as a D_{n-1} module into a direct sum of simple D_{n-1} modules $V_{n-1,\mu}$, where μ ranges over all Young diagrams obtained by removing or (if λ contains fewer than n boxes) adding a box to λ .

This means that the Bratelli diagram can be easily constructed using an inductive method. The $n = 1$ level is a single module, indexed by the Young diagram consisting of a single box. The $n = 2$ level has three modules, one indexed by the empty Young diagram, and the other two indexed by the two Young diagrams of two boxes. All three are connected to the module on the previous level.

Starting at level $n = 3$, we can construct the levels using reflections. Reflecting the $n - 2$ level of the diagram, including its connections to the $n - 1$ level, across a line drawn through the $n - 1$ level, gives the portion of the n level indexed by Young diagrams with less than n boxes. This portion of the algebra, following [BW], is given the notation H'_n . The portion indexed by Young diagrams with n boxes is then constructed as usual from the modules at the $n - 1$ level with $n - 1$ boxes in their Young diagrams. This portion of the algebra is given the notation H_n , and is isomorphic to the Hecke algebra. Consequently, all monomials in the algebra which contain a E_i factor are located in H'_n . The Bratelli diagram of C_n up to $n = 4$ is shown in Figure 2.

As with any Bratelli diagram, the dimension of each module (and its representation) is the sum of the dimensions of the representations it restricts to, which is thus the number of paths from the top of the diagram. For H_n , these dimensions can also be found from the hook length formula.

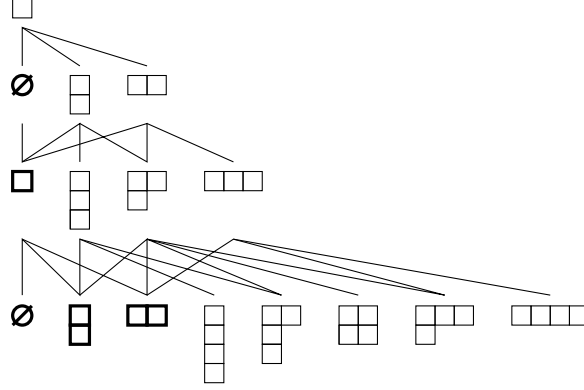


Figure 2: The Bratelli diagram for C_n (up to $n = 4$). H'_n is in bold.

3 Dimension Argument

Since we are claiming that the Krammer representation, which has dimension $\binom{n}{2}$, is an irreducible representation of C_n , it would be good to know that a representation of the appropriate size exists:

Theorem 1. *In the n th level of the Bratelli diagram of C_n , the representations labeled by rectangular Young diagrams of shapes $(n-2) \times 1$ and $1 \times (n-2)$ have dimension $\binom{n}{2}$.*

Proof. Due to the symmetric and dual nature of the Young diagrams, both arguments will be similar, and I only need discuss one. So without loss of generality, let λ denote the $(n-2) \times 1$ Young diagram, consisting of 1 row with $n-2$ boxes.

We can use the Bratelli diagram (see Figure 3) to count the dimension of each of the representations, as it will be the sum of the dimensions of the representations on the previous level that this one is connected to. Since λ has fewer than n boxes, this module is part of H'_n , and using the inductive construction of the Bratelli diagram, we can see that all the connections leading into this representation come from connections leading out of the representation $V_{n-2,\lambda} \subset H_{n-2}$ (and, in fact, go to/from the same representations on the $n-1$ level). These connections are of two types: those connecting $V_{n-2,\lambda}$ to modules in H_{n-1} , and those connecting it to modules in H'_{n-1} . The connections to H_{n-1} are easy to see, as they result from the standard Young's lattice. There is one representation with a rectangular Young diagram of

shape $(n-1) \times 1$, of dimension 1, and one representation whose Young diagram has a single box in the second row. By the hook length formula, this representation has dimension $n-2$.

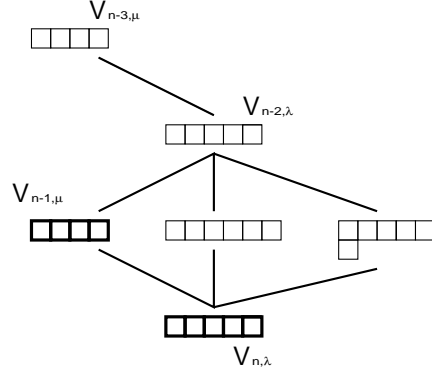


Figure 3: A portion of the Bratelli diagram.

Whatever connections exist between $V_{n-2,\lambda}$ and H'_{n-1} again come from reflections of connections from $V_{n-2,\lambda}$ to C_{n-3} . Of these, there is only one: $V_{n-2,\lambda} \subset H_{n-2}$ only connects to H_{n-3} , and since λ has shape $(n-2) \times 1$, it only connects to the representation labeled by the Young diagram μ of shape $(n-3) \times 1$. So our third and last downward connection from $V_{n-2,\lambda}$ (and by reflection, upward connection from $V_{n,\lambda}$) is to $V_{n-1,\mu}$. Notice that this is precisely the $n-1$ version of the representation we are investigating! Thus by induction, we may assume that this representation has dimension $\binom{n-1}{2}$. (One may verify that the claimed formula holds for a sufficiently low base case.)

The dimension of the representation $V_{n,\lambda}$ is therefore

$$\binom{n-1}{2} + (n-2) + 1 = \binom{n}{2}.$$

□

4 Explicit form of the representation

In Jones [J], it is shown that for any braid index n , there exists a 1-dimensional irreducible representation of the Hecke algebra. More specifically, the Hecke

algebra has a 1-dimensional invariant subspace which is preserved under multiplication by any of the Hecke algebra (or braid) generators.

It is similarly shown in [BW] and [W1] that C_n , which contains a subalgebra isomorphic to the Hecke algebra, also contains a 1-dimensional irreducible representation of C_n for any index n . This is easy to see on the Bratelli diagram as the representation $V_{n,\lambda}$ (with λ as in the proof of Theorem 2). Thus we again have a 1-dimensional invariant subspace which is preserved under multiplication by any of the braid generators. Note that this subspace does not consist of the same algebra elements as in the Hecke algebra. The quadratic relation is different if we consider the full Birman-Murakami-Wenzl algebra, so either subspace is not preserved if we use the multiplication rule corresponding to the wrong algebra.

Pick an algebra element which is in this 1-dimensional subspace (and thus generates it). We will call it v . Since the actions of the braid group on this vector (by left multiplication) result in a 1-dimensional representation, all such multiplications are scalar multiplications by the same factor, which we will call κ .

To prove that the representation of the braid group B_n given explicitly by Krammer is a representation of C_n , we will make use of the vector v which corresponds to the braid index $n - 2$. Thus, $v \in V_{n-2,\lambda} \subset H_{n-2}$ can be expressed as a linear combination of monomials in C_n using only the positive generators G_1, \dots, G_{n-3} . An explicit computation of v in these terms is possible, but not required for the computations in this paper. An inductive construction can be found in [W1]. Importantly, v has the property that

$$G_i v = \kappa v \quad \forall i < n - 2.$$

Now we will begin constructing a representation of C_n which we will later show is equivalent to Krammer's representation. A basis for the invariant subspace is the following vectors: for $1 \leq i < j \leq n$, let

$$\begin{aligned} T_{ij} &= \left(\prod_{k=i}^{j-2} G_k \right) \left(\prod_{l=j-1}^{n-1} E_l \right) v \\ &= G_i \dots G_{j-2} E_{j-1} E_j \dots E_{n-1} v. \end{aligned} \tag{13}$$

Pictorially, each vector T_{ij} corresponds to the braid-like diagram where the last two nodes on the bottom bar are connected, and the i th and j th nodes on the upper bar are connected (under all other strands), and the remaining

nodes connect from top to bottom without crossing each other. See Figure 4. (Pictures much like these are used in Jones' proof.)

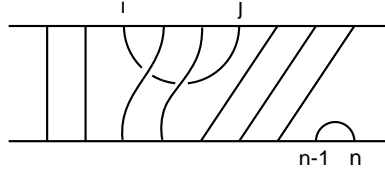


Figure 4: The basis element T_{ij}

We will occasionally ignore the restriction $i < j$ in order to write more general statements.

Now we will see how these vectors behave under a left action by the braid group. Traditionally, Artin braid generators σ_i are mapped into C_n under the map $\sigma_i \mapsto G_i$; however, that mapping turns out not to work for our purposes. We will instead rescale the braid group, and use the mapping $\sigma_i \mapsto G_i/\kappa$.

There are four different types of multiplication to consider (and three of them have subtypes which depend on the ordering of the indices):

Type A: $\sigma_i T_{i,i+1}$

Type B: $\sigma_i T_{jk}$, with $\{i, i+1\} \cap \{j, k\} = \emptyset$.

Type C: $\sigma_i T_{i+1,j}$

Type D: $\sigma_i T_{ij}$

For each of these actions, we will map $\sigma_i \mapsto G_i/\kappa$ as described above, and expand T_{ij} as defined in equation (13), and simplify the resulting expression according to the relations given in equations (1)-(12). Most of the steps are straightforward; here are lemmas for those that are less so:

Lemma 1. $E_i G_{i+1} = E_i E_{i+1} G_i^{-1}$

Proof. Both expressions are simplifications of $E_i E_{i+1} E_i G_{i+1}$. □

Lemma 2. $E_{i-2} G_i E_{i-1} E_i = E_{i-2} E_{i-1} G_{i-2} E_i$

Proof. Both expressions can simplify to $E_{i-2} G_{i-1}^{-1} E_i$. □

Now to the left action:

A.

$$\sigma_i T_{i,i+1} = \kappa^{-1} G_i (E_i E_{i+1} \dots E_{n-1} v) = \kappa^{-1} l^{-1} E_i \dots E_{n-1} v = \kappa^{-1} l^{-1} T_{i,i+1}$$

B. Because of the form of T_{jk} , the calculation will depend on the order of the indices i, j, k .

If $i + 1 < j < k$, then G_i commutes past T_{jk} to multiply by v :

$$\begin{aligned}\sigma_i T_{jk} &= \kappa^{-1} G_i (G_j \dots G_{k-2} E_{k-1} \dots E_{n-1} v) \\ &= \kappa^{-1} (G_j \dots G_{k-2} E_{k-1} \dots E_{n-1}) G_i v \\ &= \kappa^{-1} (G_j \dots G_{k-2} E_{k-1} \dots E_{n-1}) (\kappa v) = T_{jk} \quad (14)\end{aligned}$$

If $j < i < i + 1 < k$, then G_i commutes as far as it can, but it is stopped when it gets to $G_{i-1} G_i$. However, the three-term braid relation lets it transform into a G_{i-1} and continue commuting to the right:

$$\begin{aligned}\sigma_i T_{jk} &= \kappa^{-1} G_i (G_j \dots G_i \dots G_{k-2} E_{k-1} \dots E_{n-1} v) \\ &= \kappa^{-1} G_j \dots G_{i-2} G_i G_{i-1} G_i G_{i+1} \dots G_{k-2} E_{k-1} \dots E_{n-1} v \\ &= \kappa^{-1} G_j \dots G_{i-2} G_{i-1} G_i G_{i-1} G_{i+1} \dots G_{k-2} E_{k-1} \dots E_{n-1} v \\ &= \kappa^{-1} G_j \dots G_{i-2} G_{i-1} G_i G_{i+1} \dots G_{k-2} E_{k-1} \dots E_{n-1} (G_{i-1} v) \\ &= \kappa^{-1} G_j \dots G_{k-2} E_{k-1} \dots E_{n-1} (\kappa v) = T_{jk} \quad (15)\end{aligned}$$

Similarly, if $j < k < i$, then G_i commutes as far as it can, and then we use Lemma 2 to commute it the rest of the way, at the expense of lowering the index:

$$\begin{aligned}\sigma_i T_{jk} &= \kappa^{-1} G_i (G_j \dots G_{k-2} E_{k-1} \dots E_i \dots E_{n-1} v) \\ &= \kappa^{-1} G_j \dots G_{k-2} E_{k-1} \dots E_{i-2} G_i E_{i-1} E_i \dots E_{n-1} v \\ &= \kappa^{-1} G_j \dots G_{k-2} E_{k-1} \dots E_{i-2} E_{i-1} G_{i-2} E_i \dots E_{n-1} v \\ &= \kappa^{-1} G_j \dots G_{k-2} E_{k-1} \dots E_{n-1} G_{i-2} v \\ &= \kappa^{-1} G_j \dots G_{k-2} E_{k-1} \dots E_{n-1} (\kappa v) = T_{jk} \quad (16)\end{aligned}$$

C. Again, the exact calculations will depend on the order of the indices.

If $i + 1 < j$, the multiplication trivially gives us new indices for T .

$$\sigma_i T_{i+1,j} = \kappa^{-1} G_i (G_{i+1} \dots G_{j-2} E_{j-1} \dots E_{n-1} v) = \kappa^{-1} T_{ij}$$

If $j < i$, then the multiplication will behave much as in the latter parts of Case B, but with the obstruction this time located at the junction of the G terms and the E terms of $T_{j,i+1}$.

$$\begin{aligned}\sigma_i T_{j,i+1} &= \kappa^{-1} G_i (G_j \dots G_{i-1} E_i \dots E_{n-1} v) \\ &= \kappa^{-1} G_j \dots G_{i-2} G_i G_{i-1} E_i \dots E_{n-1} v \\ &= \kappa^{-1} G_j \dots G_{i-2} E_{i-1} E_i \dots E_{n-1} v = \kappa^{-1} T_{ji} \quad (17)\end{aligned}$$

D. This time, not only are the calculations dependent on the order of the indices, but the results come out differently as well.

If $i+1 < j$, then the first term in T_{ij} is G_i , so we need to use the quadratic relation in C_n .

$$\begin{aligned}\sigma_i T_{ij} &= \kappa^{-1} G_i (G_i \dots G_{j-2} E_{j-1} \dots E_{n-1} v) \\ &= \kappa^{-1} (mG_i + ml^{-1} E_i - 1) G_{i+1} \dots G_{j-2} E_{j-1} \dots E_{n-1} v \\ &= \kappa^{-1} (mT_{ij} - T_{i+1,j} + ml^{-1} E_i G_{i+1} \dots G_{j-2} E_{j-1} \dots E_{n-1} v) \quad (18)\end{aligned}$$

Two of the three summands we get are recognizable as T vectors, but the other begins with an E followed by a product of G s. For this, we need to apply Lemma 1.

$$\begin{aligned}E_i G_{i+1} G_{i+2} \dots G_{j-2} E_{j-1} \dots E_{n-1} v \\ &= E_i E_{i+1} G_i^{-1} G_{i+2} \dots G_{j-2} E_{j-1} \dots E_{n-1} v \\ &= E_i E_{i+1} G_{i+2} \dots G_{j-2} E_{j-1} \dots E_{n-1} (G_i^{-1} v) \\ &= E_i E_{i+1} G_{i+2} \dots G_{j-2} E_{j-1} \dots E_{n-1} (\kappa^{-1} v) \quad (19)\end{aligned}$$

It's still not in the standard form for a T vector, but notice what one application of Lemma 1 accomplished. We still have a word made up of E s followed by G s followed by E s, and our indices have not changed (they're still increasing by one each time). We've just changed the first G that appears into an E of the same index (and gotten a κ^{-1} scalar out of the multiplication). Repeated applications of Lemma 1 will have the same effect, and we can continue until all our G terms have been transformed into E terms. This gives us the vector $E_i \dots E_{n-1} = T_{i,i+1}$, multiplied by one κ^{-1} for each G term we had before the first application of the lemma, thus $\kappa^{-(j-2-i)}$. Conclusion:

$$\sigma_i T_{ij} = m\kappa^{-1} T_{ij} - \kappa^{-1} T_{i+1,j} + ml^{-1} \kappa^{i-j+1} T_{i,i+1}$$

If $j < i$, then things come out differently:

$$\begin{aligned}\sigma_i T_{ji} &= \kappa^{-1} G_i (G_j \dots G_{i-2} E_{i-1} \dots E_{n-1} v) \\ &= \kappa^{-1} G_j \dots G_{i-2} G_i E_{i-1} E_i \dots E_{n-1} v = \kappa^{-1} G_j \dots G_{i-2} G_{i-1}^{-1} E_i \dots E_{n-1} v \\ &= \kappa^{-1} G_j \dots G_{i-2} (m + mE_{i-1} - G_{i-1}) E_i \dots E_{n-1} v \\ &= \kappa^{-1} (mT_{ji} - T_{j,i+1} + mG_j \dots G_{i-2} E_i \dots E_{n-1} v) \quad (20)\end{aligned}$$

The last of these terms simplifies by commuting each of the G factors past all the E factors. Each G that reaches v becomes multiplication by κ , and what remains is $E_i \dots E_{n-1} v = T_{i,i+1}$. So we have

$$\begin{aligned}\sigma_i T_{ji} &= \kappa^{-1}(mT_{ji} - T_{j,i+1} + m\kappa^{i-1-j}T_{i,i+1}) \\ &= m\kappa^{-1}T_{ji} - \kappa^{-1}T_{j,i+1} + m\kappa^{i-j-2}T_{i,i+1}\end{aligned}$$

It is now clear that these vectors form an invariant subspace, of dimension $\binom{n}{2}$. As we have described an action on this space by the generators of the braid group, we have a representation of the braid group. This representation has two (complex) parameters, m and l . (Recall that κ is the eigenvalue of the specific vector v , so is not a parameter in the same sense. However, it can be adjusted or reset by rescaling, as we did above.)

A final detail is to locate this invariant subspace in C_n .

Theorem 2. *The invariant subspace described above is $V_{n,\lambda}$.*

Proof. From [W2], Prop. (1.2), it is clear that $v \in V_{n-2,\lambda}$ implies $T_{n-1,n} = E_{n-1}v \in V_{n,\lambda}$. The cited proposition, applied to the present situation, states that if $p \in V_{n-2,\lambda}$ is a minimal idempotent, then pE_{n-1} is a minimal idempotent of $V_{n,\lambda}$. For the present purposes, it is not necessary to deal with minimal idempotents, but notice that since $V_{n-2,\lambda}$ is 1-dimensional, the minimal idempotent p is a scalar multiple of v . Therefore, the cited proposition guarantees that a certain scalar multiple of vE_{n-1} is a minimal idempotent of $V_{n,\lambda}$, which is a stronger result than we require. (Notice also that since v is written using only the generators G_1, \dots, G_{n-3} , it commutes with v .) \square

5 The Krammer representation

We will now recall Krammer's representation from [K], written in terms of actions on a module with basis v_{ij} , with $1 \leq i, j \leq n$ and $i \neq j$. Like the representation above, it has two (complex) parameters, q and t . (Stephen Bigelow has found a fascinating interpretation of these parameters from a topological perspective.) Krammer's presentation is longer than that described here, because he included formulae for multiplication by band generators (a larger set than Artin generators), and because I am taking the liberty to combine

formulae when the order of the indices does not matter.

$$\begin{aligned}
\sigma_i v_{i,i+1} &= tq^2 v_{i,i+1} \\
\sigma_i v_{jk} &= v_{jk} && \text{for } \{i, i+1\} \cap \{j, k\} = \emptyset \\
\sigma_i v_{i+1,j} &= v_{ij} && \text{for } j \neq i, i+1 \\
\sigma_i v_{ij} &= tq(q-1)v_{i,i+1} + (1-q)v_{ij} + qv_{i+1,j} && \text{if } i+1 < j \\
\sigma_i v_{ji} &= (1-q)v_{ji} + qv_{j,i+1} + q(q-1)v_{i,i+1} && \text{if } j < i
\end{aligned}$$

To show that this is the same as the representation of C_n constructed above, we will rescale the basis slightly and set a correspondence between parameters:

Theorem 3. *Under the identifications $q = -\kappa^{-2}$, $m = \kappa(1-q)$, $l^{-1} = \kappa tq^2$, $v_{ij} = \kappa^{i+j} T_{ij}$, the two actions described above are identical.*

Proof. A.

$$\sigma_i v_{i,i+1} = \kappa^{2i+1} \sigma_i T_{i,i+1} = \kappa^{2i} l^{-1} T_{i,i+1} = \kappa^{2i+1} tq^2 T_{i,i+1} = tq^2 v_{i,i+1}$$

B.

$$\sigma_i v_{jk} = \kappa^{j+k} \sigma_i T_{jk} = \kappa^{j+k} T_{jk} = v_{jk}$$

C.

$$\sigma_i v_{i+1,j} = \kappa^{i+j+1} \sigma_i T_{i+1,j} = \kappa^{i+j} T_{ij} = v_{ij}$$

D. If $i+1 < j$,

$$\begin{aligned}
\sigma_i v_{ij} &= \kappa^{i+j} \sigma_i T_{ij} = m\kappa^{i+j-1} T_{ij} - \kappa^{i+j-1} T_{i+1,j} + ml^{-1} \kappa^{2i+1} T_{i,i+1} \\
&= (1-q)\kappa^{i+j} T_{ij} - \kappa^{i+j-1} T_{i+1,j} + \kappa^2(1-q)tq^2 \kappa^{2i+1} T_{i,i+1} \\
&= (1-q)v_{ij} - \kappa^{-2} v_{i+1,j} + \kappa^2(1-q)tq^2 v_{i,i+1} = (1-q)v_{ij} + qv_{i+1,j} + tq(q-1)v_{i,i+1}
\end{aligned} \tag{21}$$

If $j < i$,

$$\begin{aligned}
\sigma_i v_{ji} &= \kappa^{i+j} \sigma_i T_{ji} = m\kappa^{i+j-1} T_{ji} - \kappa^{i+j-1} T_{j,i+1} + m\kappa^{2i-2} T_{i,i+1} \\
&= (1-q)\kappa^{i+j} T_{ji} - \kappa^{i+j-1} T_{j,i+1} + (1-q)\kappa^{2i-1} T_{i,i+1} \\
&= (1-q)v_{ji} - \kappa^{-2} v_{j,i+1} + \kappa^{-2}(1-q)v_{i,i+1} = (1-q)v_{ji} + qv_{j,i+1} + q(q-1)v_{i,i+1}
\end{aligned} \tag{22}$$

□

Remark. The computations in the proof above will actually work under any additional rescaling of v_{ij} with respect to T_{ij} , namely using the identification $v_{ij} = \kappa^{i+j+k}T_{ij}$, for any value of k . For simplicity in the proof, I set $k = 0$, but if these identifications are to be used for any explicit calculations, I recommend instead using $k = n + 1$, so that the values of the exponent range symmetrically from $-(n - 2)$ to $(n - 2)$.

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